Numerical Algorithm for Solving Optimal Control Problems with Mixed Constraints

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Abstract: In this research, the numerical solutions of optimal control problems constrained by ordinary differential equation and integral equation are examined. We obtained the numerical solution by applying the "first discretize then optimize" technique. The discretization of the objective function, differential and integral constraints was done using trapezoidal rule, $\frac{1}{3}$ Simpson's rule and fourth-order Adams-Moulton respectively. Thereafter, the formulated constrained optimization problem was converted into unconstrained problem by applying augmented lagrangian functional. We finally applied the Quasi-Newton algorithm of the Broydon-Fletcher-Goldfrab-Shannon (BFGS) type to obtain our optimal solution. Two examples of optimal control problems constrained by ordinary differential equation and an integral equation are considered. We obtained promising results with linear convergence.

Keywords: Optimal Control Problem, Optimization theory, Algorithms, Discretization, Quasi-Newton, Augmented Lagrangian, Constrained OCP, Unconstrained OCP, Numerical Solution of OCP, Mixed Constraints

1. Introduction

Optimal control problems are formulated mathematical models involving state and control variables. The application of optimal control theory is used in minimizing a given objective function by getting the control law of the constraints for a given period of time (t). The knowledge of optimal control theory is applicable in engineering, science, technology, manufacturing companies, epidemiology, modeling, etc. It is worthy to note that optimal control theory evolved from calculus of variations as shown in [5].

The general form of an optimal control problem (OCP) with differential and integral constraints is given as:

Minimize:

$$J(t, x(t), u(t)) = \int_{t_0}^{t_f} f(t, x(t), u(t))dt \tag{1}$$

Subject to:

$$\dot{x}(t) = g(t, x(t), u(t)) \qquad t_0 \leq t \leq t_f \tag{2}$$

$$y(t) = \int_{t_0}^{t_f} s(t, x(t), u(t))dt \tag{3}$$

$$x(t_0) = x_0 \tag{4}$$

Where $x \in R^n$ and $u \in R^m$ are the state and control variables defined in the domain $t \in [t_0, t_f]$ and f: $[t_0, t_f] \times R^n \times R^m \rightarrow R$, g: $[t_0, t_f] \times R^n \times R^m \rightarrow R^n$ and s: $[t_0, t_f] \times R^n \times R^m \rightarrow R$.

It is also possible to have an OCP without any constraint and still get an optimal solution to the problem as shown by [11]. Since (OCPs) are always in continuous-time, it is important to convert it to discrete problem using appropriate numerical schemes. The set of state variables is given by $(t, x(t) \in X)$ and the set of the control variables is given by $(t, u(t) \in U)$ in a given time t with $t_0 \leq t \leq t_f$. [1] demonstrated how to get analytical solution of an optimal control problem (OCP). The work of [2] and [4] shows the numerical process of finding numerical solution to OCP and [3] provided numerical solution to an OCP with delay that cannot be solved analytically. An OCP can also have vector and matrix coefficients as we have in the work of [6]. [4] worked on OCP constrained by an integral equation. As optimal control theory expands, it is important to consider problems with more than one constraints. [3], [6], and [8] all considered optimal control problems with more than one constraints, however, the mixed constraints are only of equality and inequality type. Hence, the need to extend our research with OCP constrained by differential and integral constraints. [9] combined Augmented Lagrangian and penalty methods to solve the special kind of OCP considered. [13] developed a new augmented Lagrangian function from a new multiplier method to solve nonlinear OCP with equality and inequality constraints.

2. Methodology

Given the optimal control problem with mixed integral-differential constraints:

Minimize:

$$J(t, x(t), u(t)) = \int_{t_0}^{t_f} (px^2(t) + qu^2(t))dt \tag{5}$$

Subject to:

$$\dot{x}(t) = ax(t) + bu(t) \qquad t_0 \leq t \leq t_f \tag{6}$$

$$y(t) = \int_{t_0}^{t_f} (cx(t) + du(t))dt \tag{7}$$

$$x(t_0) = x_0 \tag{8}$$

Where $p, q, a, b, c, d \in R$, and $p, q > 0$ and $t \in [t_0, t_f]$

Discretization of the objective function:

$$Min\ J(t, x(t), u(t)) = \int_{t_0}^{t_f} (px^2(t) + qu^2(t))dt$$

The discretization of J (t, x(t), u(t)) (5) was carried out using the Trapezoidal rule of the form:

$$\int_{t_0}^{t_f} f(x)dx \simeq \frac{h}{2} \sum_{k=0}^{N-1} (f_k + f_{k+1}) \tag{9}$$

Applying equation (9) on (5):

$$J(t, x(t), u(t)) = \int_{t_0}^{tf} (px^2(t) + qu^2(t))\, dt$$

$$J(t, x(t), u(t)) = \int_{t_0}^{tf} (px^2(t)dt) + \int_{t_0}^{tf} (qu^2(t)dt)$$

Now, $\int_{t_0}^{t_f} px^2(t)dt \simeq p\frac{h}{2} \sum_{k=0}^{N-1} (f_k + f_{k+1})$

$\simeq \frac{ph}{2} \sum_{k=0}^{N-1} f_k + \frac{ph}{2} \sum_{k=0}^{N-1} f_{k+1}$

$\simeq \frac{ph}{2} \sum_{k=0}^{N-1} x_k^2 + \frac{ph}{2} \sum_{k=0}^{N-1} x_{k+1}^2$

$\simeq \frac{ph}{2}(x_0^2 + x_1^2 + x_2^2 + x_3^2 + \cdots + x_{N-1}^2) + \frac{ph}{2}(x_1^2 + x_2^2 + x_3^2 + \cdots + x_{N-1}^2 + x_N^2)$

$\simeq (\frac{ph}{2}x_0^2 + phx_1^2 + phx_2^2 + phx_3^2 + \cdots + phx_{N-1}^2 + \frac{ph}{2}x_N^2)$

If $L = \frac{ph}{2}$

$\int_{t_0}^{t_f} px^2(t)dt = (Lx_0^2 + 2Lx_1^2 + 2Lx_2^2 + 2Lx_3^2 + \cdots + 2Lx_{N-1}^2 + Lx_N^2)$

Which can be represented in matrix form as:

$$\int_{t_0}^{t_f} px^2(t)dt = \begin{pmatrix} x_1, & x_2, & \ldots, & x_N \end{pmatrix} \begin{bmatrix} 2L & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 2L & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 2L & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 2L & 0 \\ 0 & 0 & 0 & . & . & . & 0 & L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_{N-1} \\ x_N \end{bmatrix} + Lc_0^2$$

Therefore:

$$A = \begin{bmatrix} 2L & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 2L & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 2L & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 2L & 0 \\ 0 & 0 & 0 & . & . & . & 0 & L \end{bmatrix}$$

$$\int_{t_0}^{t_f} px^2(t)dt = X^T AX + C \qquad (9a)$$

Where $x_0^2 = c_0^2$(constant)

X is $n-$dimensional column vector

$A = [a_{ij}]$ is a $n \times n$ diagonal matrix *i.e.,* $A \in R^{N,N}$ which is compactly defined as:

$$A = [a_{ij}] = \begin{cases} L, & i = j = N \\ 2L, & j = i \quad 1 \le i \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

$$C = Lc_0^2$$

Similarly,

$$\int_{t_0}^{t_f} qu^2(t)dt \simeq q(\tfrac{h}{2}\sum_{k=0}^{N-1}(f_k + f_{k+1}))$$
$$\simeq \tfrac{qh}{2}\sum_{k=0}^{N-1} f_k + \tfrac{qh}{2}\sum_{k=0}^{N-1} f_{k+1}$$
$$\simeq \tfrac{qh}{2}\{u_0^2 + u_1^2 + u_2^2 + u_3^2 + \cdots + u_{N-1}^2\} + \tfrac{qh}{2}\{u_1^2 + u_2^2 + u_3^2 + \cdots + u_{N-1}^2 + u_N^2\}$$
$$\simeq \{\tfrac{qh}{2}u_0^2 + qhu_1^2 + qhu_2^2 + qhu_3^2 + \cdots + qhu_{N-1}^2 + \tfrac{qh}{2}u_N^2\}$$

If $M = \frac{qh}{2}$

$$\int_{t_0}^{f} qu^2(t)dt = \{Mu_0^2 + 2Mu_1^2 + 2Mu_2^2 + 2Mu_3^2 + \cdots + 2Mu_{N-1}^2 + Mu_N^2\}.$$

This can be represented in matrix form as:

$$\int_{t_0}^{t_f} qu^2(t)dt = \begin{pmatrix} u_0, & u_1, & \ldots & u_{N-1}, & u_N \end{pmatrix} \begin{bmatrix} M & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 2M & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 2M & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 2M & 0 \\ 0 & 0 & 0 & . & . & . & 0 & M \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ . \\ . \\ . \\ u_{N-1} \\ u_N \end{bmatrix}$$

Therefore:

$$B = \begin{bmatrix} M & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 2M & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 2M & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 2M & 0 \\ 0 & 0 & 0 & . & . & . & 0 & M \end{bmatrix}$$

$$\int_{t_0}^{t_f} qu^2(t)dt = U^T BU \qquad (9b)$$

Where: u is $(n+1)$ dimensional column vector,

$B = [b_{ij}]$ is a $(N+1) \times (N+1)$ diagonal matrix *i.e.,* $B \in R^{N+1, N+1}$ which is compactly defined as:

$$B = [b_{ij}] = \begin{cases} M, & i = j = 1, N+1 \\ 2M, & j = i, \quad 2 \le i \le N \\ 0, & \text{elsewhere} \end{cases}$$

Therefore,

$$\int_{t_0}^{t_f} (px^2(t) + qu^2(t))dt = \{\tfrac{h}{2}[px_0^2 + qu_0^2 + px_N^2 + qu_N^2] + h\sum_{k=1}^{N-1}[x_k^2 + u_k^2]\}$$
$$\int_{t_0}^{t_f} (px^2(t) + qu^2(t))dt = Z^T(A|B)Z + C = Z^T WZ + C$$

Where:

$$Z = \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_N & u_0 & u_1 & u_2 & \ldots & u_N \end{pmatrix}^T$$

is a $(2n+1) \times 1$ diagonal vector of x and u.

The Matrices A and B are augmented into a sparse real symmetric positive definite quadratic operator of dimension $(2n+1) \times (2n+1)$ expressed below as:

$$W=[W_{ij}]=\begin{bmatrix} 2L & 0 & 0 & . & . & . & 0 & 0 & . & . & . & . & . & 0 \\ 0 & 2L & 0 & . & . & . & . & . & . & . & . & . & . & 0 \\ 0 & 0 & 2L & . & . & . & . & . & . & . & . & . & . & 0 \\ . & 0 & 0 & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & L & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & . & M & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & . & . & 2M & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & 2M & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & . & . & . & . & . & . & 2M & 0 \\ 0 & 0 & 0 & . & . & . & . & . & . & . & . & . & . & M \end{bmatrix}+Lc_0^2$$

W is defined compactly as:

$$W=[w_{ij}]=\begin{cases} L, & i=j=N \\ M, & j=i=N+1, 2N+1 \\ 2L, & j=i \text{ for } \ 1\leq i\leq N-1 \\ 2M, & j=i \text{ for } \ N+2\leq i\leq 2N \\ 0, & \text{elsewhere} \end{cases}$$

Discretization of the differential constraint.

Using the $\frac{1}{3}$ Simpson's rule numerical for this discretization given as:

$$\begin{aligned} x_{k+2}-x_k &= \frac{h}{3}[f_{k+2}+4f_{k+1}+f_k]+O(h^4) && (10)\\ x_{k+2}-x_k &= \frac{h}{3}[(ax_{k+2}+bu_{k+2})+4(ax_{k+1}+bu_{k+1})+(ax_k+bu_k)] && (11)\\ x_{k+2} &= \frac{h}{3}[ax_{k+2}+bu_{k+2}+4ax_{k+1}+4bu_{k+1}+ax_k+bu_k]+x_k && (12)\\ x_{k+2}-\frac{ah}{3}x_{k+2} &= \frac{bh}{3}u_{k+2}+\frac{4ah}{3}x_{k+1}+\frac{4bh}{3}u_{k+1}+\frac{ah}{3}x_k+x_k+\frac{bh}{3}u_k && (13)\\ (1-\frac{ah}{3})x_{k+2} &= \frac{bh}{3}u_{k+2}+\frac{4ah}{3}x_{k+1}+\frac{4bh}{3}u_{k+1}+\frac{bh}{3}u_k+(1+\frac{ah}{3})x_k && (14)\\ (\frac{3-ah}{3})x_{k+2} &= \frac{bh}{3}u_{k+2}+\frac{4ah}{3}x_{k+1}+\frac{4bh}{3}u_{k+1}+\frac{bh}{3}u_k+(\frac{3+ah}{3})x_k && (15) \end{aligned}$$

Multiply through by $(\frac{3}{3-ah})$,

$$\begin{aligned} x_{k+2} &= -(\frac{ah+3}{ah-3})x_k-(\frac{4ah}{ah-3})x_{k+1}-(\frac{bh}{ah-3})u_{k+2}-(\frac{4bh}{ah-3})u_{k+1} \\ &\quad -(\frac{bh}{ah-3})u_k \end{aligned} \qquad (16)$$

If: $R=(\frac{ah+3}{ah-3}),\ S=(\frac{4ah}{ah-3})$, and $T=(\frac{bh}{ah-3})$, Then:

$x_{k+2} = -Rx_k - Sx_{k+1} - Tu_{k+2} - 4Tu_{k+1} - Tu_k$ (17)
$Rx_k + Sx_{k+1} + x_{k+2} + Tu_k + 4Tu_{k+1} + Tu_{k+2} = 0.$ (18)
Setting $k = 0$;
$Sx_1 + x_2 + T[u_0 + 4u_1 + u_2] = -Rx_0.$ (19)
$Rx_1 + Sx_2 + x_3 + T[u_1 + 4u_2 + u_3] = 0.$ (20)
Setting $k = 2$;
$Rx_2 + Sx_3 + x_4 + T[u_2 + 4u_3 + u_4] = 0.$ (21)
$\vdots$
Setting $k = N - 2$;
$Rx_{N-2} + Sx_{N-1} + X_N + T[u_{N-2} + 4u_{N-1} + u_N] = 0.$ (22)

This can be represented in matrix form as:

$$\left(\begin{array}{cccccccccc|cccccccccc} S & 1 & 0 & 0 & . & . & . & 0 & 0 & & T & 4T & T & 0 & . & . & . & 0 & 0 \\ R & S & 1 & 0 & . & . & . & 0 & 0 & & 0 & T & 4T & T & . & . & . & 0 & 0 \\ 0 & R & S & 1 & . & . & . & 0 & 0 & & 0 & 0 & T & 4T & T & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 & 0 & & 0 & 0 & 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & R & S & 1 & & 0 & 0 & 0 & . & . & . & T & 4T & T \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_N \\ u_0 \\ u_1 \\ u_2 \\ . \\ . \\ . \\ u_N \end{pmatrix} = \begin{bmatrix} -Rx_0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix}$$

This implies that:

$(F|G)Z = JZ = H$ (23)

Where F is a $(N-1) \times N$ matrix and G is a $(N-1) \times (N+1)$ matrix.

Hence, $(F|G) = J$ is an $(N-1) \times (2N+1)$ matrix compactly defined as:

$$J = [j_{ij}] = \begin{cases} S, & 1 \leq i \leq N-1, \quad \text{where } j = i \\ 1, & 1 \leq i \leq N-1, \quad j = i+1 \\ R, & 2 \leq i \leq N-1, \quad j = i-1 \\ T, & 1 \leq i \leq N-1, \quad j = N+i, \ N+i+2 \\ 4T, & 1 \leq i \leq N-1, \quad j = N+i+1 \\ 0, & \text{elsewhere} \end{cases}$$

$Z = (x_1, x_2, ., ., ., x_N, u_0, u_1, ., ., ., u_N)$ and H is a column vector.

Discretization of the integral constraint in equation (7)

$$y(t) = \int_{t_0}^{t_f} (cx(t) + du(t))dt$$

The discretization was carried out using fourth-order Adams-Moulton numerical scheme which is given as:

$$y_{i+1} = y_i + \frac{h}{24}[9f(t_{i+1}, y_{i+1}) + 19f(t_i, y_i) - 5f(t_{i-1}, y_{i-1}) + f(t_{i-2}, y_{i-2})] \quad (24)$$

Applying on the integral constraint gives:

$$x_{i+1} = x_i + \frac{h}{24}[9(cx_{i+1}+du_{i+1})+19(cx_i+du_i)-5(cx_{i-1}+du_{i-1})+(cx_{i-2}+du_{i-2})] \quad (25)$$

$$x_{i+1} = x_i + \frac{9h}{24}cx_{i+1} + \frac{9h}{24}du_{i+1} + \frac{19h}{24}cx_i + \frac{19h}{24}du_i - \frac{5h}{24}cx_{i-1} - \frac{5h}{24}du_{i-1} + \frac{h}{24}cx_{i-2} + \frac{h}{24}du_{i-2} \quad (26)$$

Collect like terms:

$$x_{i+1} - \frac{9h}{24}cx_{i+1} = x_i + \frac{19h}{24}cx_i - \frac{5h}{24}cx_{i-1} + \frac{h}{24}cx_{i-2} + \frac{9h}{24}du_{i+1} + \frac{19h}{24}du_i - \frac{5h}{24}du_{i-1} + \frac{h}{24}du_{i-2} \quad (27)$$

$$(\frac{24-9hc}{24})x_{i+1} = (\frac{24+19hc}{24})x_i - \frac{5h}{24}cx_{i-1} + \frac{h}{24}cx_{i-2} + \frac{9h}{24}du_{i+1} + \frac{19h}{24}du_i - \frac{5h}{24}du_{i-1} + \frac{h}{24}du_{i-2} \quad (28)$$

Multiply through by $(\frac{24}{24-9hc})$

$$x_{i+1} = (\frac{24+19hc}{24-9hc})x_i - \frac{5hc}{24-9hc}x_{i-1} + \frac{hc}{24-9hc}x_{i-2} + \frac{9hd}{24-9hc}u_{i+1} + \frac{19hd}{24-9hc}u_i - \frac{5hd}{24-9hc}u_{i-1} + \frac{hd}{24-9hc}u_{i-2} \quad (29)$$

If $A = \frac{24+19hc}{24-9hc}$, $B = -\frac{5hc}{24-9hc}$, $D = \frac{hc}{24-9hc}$ $E = \frac{9hd}{24-9hc}$, $I = \frac{19hd}{24-9hc}$, $K = \frac{5hd}{24-9hc}$ and $V = \frac{hd}{24-9hc}$

Hence, we have:

$$x_{i+1} = Ax_i + Bx_{i-1} + Dx_{i-2} + Eu_{i+1} + Iu_i + Ku_{i-1} + Vu_{i-2} \quad (30)$$

$$x_{i+1} - Ax_i - Bx_{i-1} - Dx_{i-2} - Vu_{i-2} - Ku_{i-1} - Iu_i - Eu_{i+1} = 0 \quad (31)$$

Setting $i = 2$;

$$x_3 - Ax_2 - Bx_1 - Vu_0 - Ku_1 - Iu_2 - Eu_3 = Dx_0 \quad (32)$$

Setting $i = 3$;

$$x_4 - Ax_3 - Bx_2 - Dx_1 - Vu_1 - Ku_2 - Iu_3 - Eu_4 = 0 \quad (33)$$

⋮

Setting $i = N-3$;

$$x_{N-2} - Ax_{N-3} - Bx_{N-4} - Dx_{N-5} - Vu_{N-5} - Ku_{N-4} - Iu_{N-3} - Eu_{N-2} = 0 \quad (34)$$

Setting $i = N-1$;

$$x_N - Ax_{N-1} - Bx_{N-2} - Dx_{N-3} - V u_{N-3} - Ku_{N-2} - Iu_{N-1} - Eu_N = 0 \quad (35)$$

This can be represented in matrix form as:

$$\left(\begin{array}{ccccccccc|ccccccccc} -B & -A & 1 & 0 & . & . & . & 0 & 0 & -V & -K & -I & -E & . & . & . & 0 & 0 \\ -D & -B & -A & 1 & . & . & . & 0 & 0 & 0 & -V & -K & -I & -E & . & . & . & 0 \\ 0 & -D & -B & -A & 1 & . & . & . & 0 & 0 & 0 & -V & -K & -I & -E & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & -D & -B & -A & 1 & 0 & 0 & . & . & . & -V & -K & -I & -E \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_N \\ u_0 \\ u_1 \\ u_2 \\ . \\ . \\ . \\ u_N \end{pmatrix} = \begin{bmatrix} Dx_0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix}$$

This implies that:

$(O|K)Z = PZ = Q$ (36)

Where $(O|K) = P$ is an $(N-2) \times (2N+1)$ matrix compactly defined as:

$$P = [P_{ij}] = \begin{cases} -B, & 1 \le i \le n-2 \quad \text{where } j = i \\ -A, & 1 \le i \le n-2 \quad j = i+1 \\ -D, & 2 \le i \le n-2 \quad j = i-1 \\ 1, & 1 \le i \le n-2 \quad j = i+2 \\ -V, & 1 \le i \le n-2 \quad j = i \\ -K, & 1 \le i \le n-2 \quad j = i+1 \\ -I, & 1 \le i \le n-2 \quad j = i+2 \\ -E, & 1 \le i \le n-2 \quad j = i+3 \\ 0, & \text{elsewhere} \end{cases}$$

$Z = (x_1, x_2, x_N, u_0, u_1, u_N)^T$ is a column vector and Q is on the right hand side it is also a column vector.

The discretized optimal control problem becomes a large sparse quadratic programming problem which is written as:

Minimize:

$$f(Z) = Z^T W Z + C \tag{37}$$

Subject to:

$$JZ = H, \tag{38}$$

$$PZ = Q \tag{39}$$

The Numerical Algorithm for Solving One-Dimensional Optimal Control Problems with Mixed Integral-Differential Constraints is given as:

(1) **Compute the given variables W, J, H**

(2) **Choose** $Z_{0,0} \in \mathbb{R}^{n \times (n+1)}$, $B_0 = I$, T^* **(Tolerance) and Initialize** $\mu_{1k} > 0$, $\lambda_{1k} > 0$, $\mu_{2k} > 0$, $\lambda_{2k} > 0$ **by setting** $k = 0$

(3a) **Set** $i = 0$ **and** $g_0 = \nabla L(Z_{0,0}) = \nabla L_0$

(3b) **Compute** $W_i = [W + \frac{\mu_{1k}}{2} J^T J + \frac{\mu_{2k}}{2} P^T P]$, $J_i = [\lambda_{1k}^T J + \lambda_{2k}^T P - \mu_{1k} H^T J - \mu_{2k} Q^T P]$, $H_i = [\frac{\mu_{1k}}{2} H^T H + \frac{\mu_{2k}}{2} Q^T Q - \lambda_{1k}^T H - \lambda_{2k}^T Q + C]$

(3c) **Set** $P_i = -[B_i]g_i$ **(search direction) and**

(3d) **Compute** $\alpha_i^* = \frac{-(J_i P_i + Z_i^T W_i P_i)}{P_i^T W_i P_i}$ **(Step length)**

(3e) **Set** $Z_{k,i+1} = Z_{k,i} + \alpha_{1i} P_i + \alpha_{2i} P_i$ **and**

(3f) **Compute** $g_{i+1} = \nabla(Z_{k,i+1}, \lambda_{1k}, \mu_{1k}, \lambda_{2k}, \mu_{2k})$

(3g) **If** $||\nabla(Z_{k,i+1}, \lambda_{1k}, \mu_{1k}, \lambda_{2k}, \mu_{2k})|| \leq T^*$ **go to Step (4) Else go to (3h)**

(3h) **Set** $q_i = g_{i+1} - g_i$ **and** $P_i = Z_{i+1} - Z_i$

3(i) **Compute** $B_i^u = [1 + \frac{q_i^T B_i q_i}{P_i^T q_i}][\frac{P_i P_i^T}{P_i^T q_i}] - [\frac{(P_i q_i^T B_i) + (B_i q_i P_i^T)}{P_i^T q_i}]$

(3j) **Set** $B_{i+1} = B_i + B_i^u$ **and repeat steps** $(3a) - (3f)$ **for next** $i = i + 1$

(4) **If** $||JZ_{k,i+1} - H|| \leq T^*$ **Stop! Choose** $Z_{k,i+1}$ **and compute** $W_{k,i+1}^*$ **Else go to step (5)**

(5) **Update** $\mu_{j(k+1)} = \mu_{j0} \times 2^{k+1}$ **(Penalty) and** $\lambda_{j(k+1)} = \lambda_{jk} + \mu_{jk}(JZ_k - H)$ **(Multiplier) for** $j = 1, 2$

(6) **Go to step (3) for next** $k = k + 1$

3. Results

Problem I:

Consider a one-dimensional optimal control problem by mixed integral-differential constraints given as:

$$Min\ J(t, x(t), u(t)) = \int_0^1 (2x^2(t) + u^2(t))dt \quad (40)$$

$$s.t:\ \dot{x}(t) = 2x(t) + u(t), \quad 0 \leq t \leq 1 \quad x(0) = 1 \quad (41)$$

$$x(t) = \int_0^1 (x(t) + u(t))dt \quad (42)$$

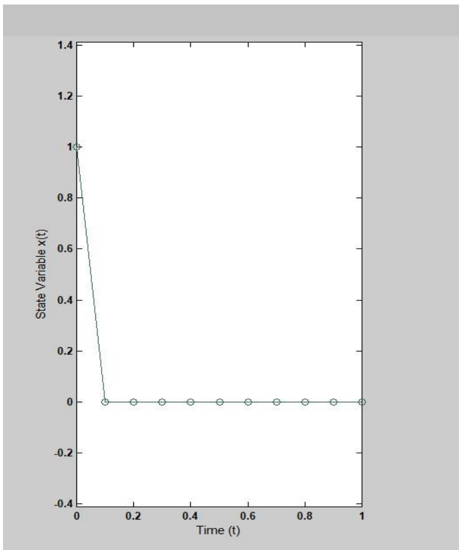
From the problem above, it is obvious that: $p = 2$, $q = 1$, $a = 2$, $b = 1$, $c = 1$, $d = 1$, $x_0 = 1$ We obtained the optimal objective value from the Quasi-Newton Embedded with Augmented Lagrangian function from MATLAB as 3.066210015 where $h = 0.2$, $\mu = 10^6$, $Tol = 10^{-5}$

Table I: Convergence Analysis of Problem I

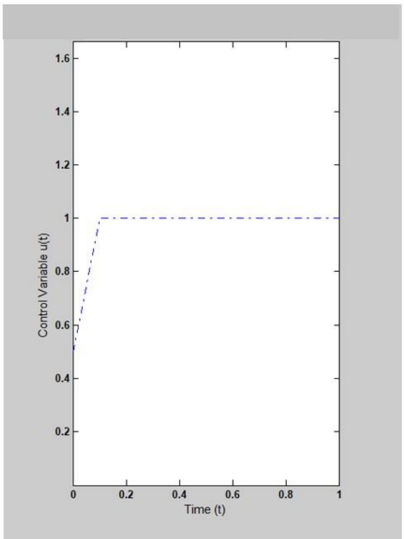
Penalty Parameter (μ)	Objective Value (J)	Convergence ratio (β)	Tol.
1.0×10^2	4.026960214	0.147330515	10−5
1.0×10^3	3.838207687	0.102953394	10−5

1.0×10^4	3.633857231	0.018234309	10−5
1.0×10^5	3.365485875	0.000125793	10−5
1.0×10^6	3.065738162	0.000000167	10−5

The result from convergence ratio shows that the developed algorithm has linear convergence since it is between 0 and 1.



(i) Graphical plot of the state variable (x) against Time (t)



(ii) Graphical plot of the control variable (u) against Time (t)

Figure 1: Graphical solution of optimal control problem with integral and differential constraints with real coefficients.

Problem II:

Consider a one-dimensional optimal control problem by mixed integral-differential constraints given as:

$$Min\ J(t, x(t), u(t)) = \int_0^5 (3x^2(t) + 2u^2(t))dt \quad (43)$$

$$s.t : \dot{x}(t) = x(t) + 2u(t), \qquad 0 \leq t \leq 1 \qquad x(0) = -1 \quad (44)$$

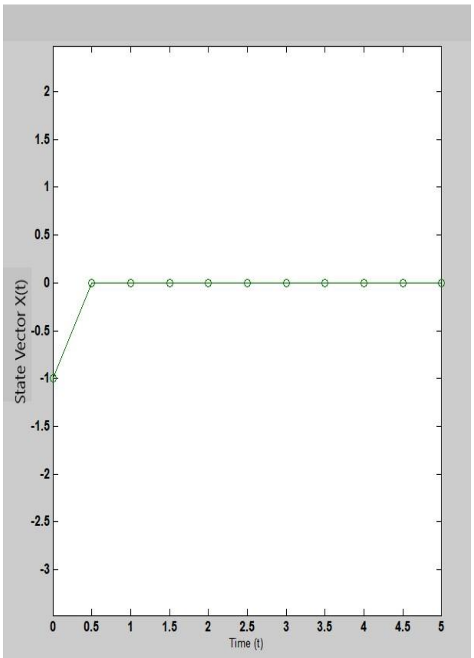
$$x(t) = \int_0^5 (2x(t) + u(t))dt \quad (45)$$

From the problem above, it is obvious that: p = 3, q = 2, a = 1, b = 2, c = 2, d = 1, x0 = 1 We obtained the optimal objective value from the Quasi-Newton Embedded with Augmented Lagrangian function from MATLAB as 1.943752127 where h = 0.5, μ = 106, Tol = 10−5

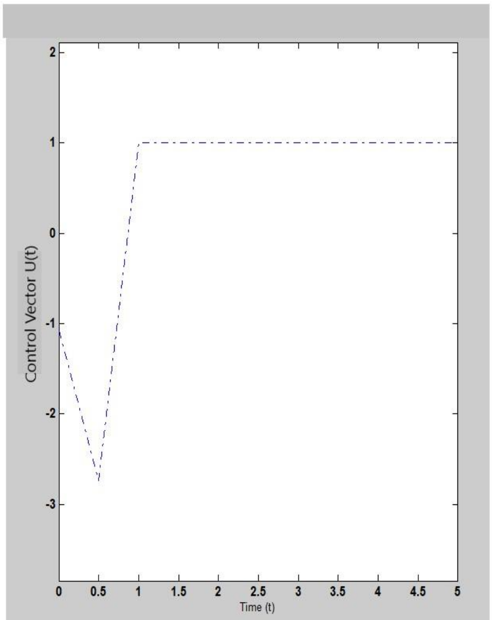
Table II: Convergence Analysis of Problem II

Penalty Parameter (μ)	Objective Value (J)	Convergence ratio (β)	Tol.
1.0×10^2	2.620818465	0.194801175	10−5
1.0×10^3	2.481767271	0.146823271	10−5
1.0×10^4	2.334670058	0.117336358	10−5
1.0×10^5	2.174695771	0.000147362	10−5
1.0×10^6	1.943752127	0.000000115	10−5

We obtained an optimal solution to this problem with linear convergence.



(i) Graphical plot of the state variable (x) against Time (t)



(ii) Graphical plot of the control variable (u) against Time (t)

Figure 2: Graphical solution of optimal control problem with integral and differential constraints with real coefficients.

4. Conclusion

Quasi-Newton Embedded with Augmented Lagrangian is applied to continuous optimization problems with mixed integral-differential constraints; we developed a unique scheme for a one-dimensional optimal control problem considered. The graph of the state and control variables is plotted against time to show the behavior of the variables with respect to time. The convergence ratio from the two examples considered shows that the newly-developed algorithm has linear convergence.

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