PLANETARY EQUATION IN THE EQUATORIAL PLANE OF A HOMOGENEOUS PROLATE SPHEROIDAL BODY ACCORDING TO EINSTEIN'S GEOMETRICAL LAWS OF GENERAL RELATIVITY

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Abstract: The solution Einstein's geometrical equation of motion for test particles of non-zero rest masses in gravitational field in spherical polar coordinates (r, θ, ϕ) was first obtained by Karl Schwarzchild, and the result were confined to spherical body only along the origin. But the earth and other planets as well as the sum are now well-known experimentally as spheroidal in shape. Therefore, the need arises to extend the Einstein's. geometrical equation of motion for test particles in gravitation field of bodies of perfect spherical geometry to those of spheroidal geometry. In this paper, we are out to derive the planetary equation in the equatorial plane of a homogeneous prolate spheroidal massive body according to Einstein's geometrical law of general relativity expressed in terms of prolate spheroidal coordinates. The coefficients of affine connection for the gravitational field are used to derive the equations of motion in the equatorial plane for the test particles. Then we show how they may be solved exactly and analytically for motion confined to the equatorial plane for mathematical investigations and hence physical interpretation and experimental investigations as compared with the other well-known results for bodies in the solar system.

Keywords: Prolate Spheroidal Coordinates, Einstein's Geometrical Laws, Schwartzchild's Metric Tensor and Planetary Equations.

1:0 Introduction

The formulation and solution of Einstein's Geometrical Equation of motion for test particles of non-zero rest masses in the gravitational fields for massive bodies is well-known [1]. These equations were first obtained by Karl Schwarzchild, popularly known as the Schwarzchild's Geodesic Equations. They are given in the Spherical Polar Coordinates (r, θ, ϕ)as:

$$\ddot{t} + \frac{2k}{c^2r^2}\left(1 - \frac{2k}{c^2r}\right)\dot{t}\dot{r} = 0 \quad (1)$$

$$\ddot{r} + \frac{k}{r^2}\left(1 - \frac{2k}{c^2r}\right)\dot{t}^2 - \frac{k}{c^2r^2}\left(1 - \frac{2k}{c^2r}\right)^{-1}\dot{r}^2 - r\left(1 - \frac{2k}{c^2r}\right)\dot{\theta}^2 - r\left(1 - \frac{2k}{c^2r}\right)sin^2\theta\dot{\phi}^2 = 0 \quad (2)$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - sin\theta cos\theta\dot{\phi}^2 = 0 \quad (3)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2cot\dot{\theta}\dot{\phi} = 0 \quad (4)$$

and

$$k = GM_0 (5)$$

whereM_0 is the rest mass of the body, c is the speed of light, r is the radius of the earth, t is the time coordinate, Θ is the polar angle, ϕ is the azimuthal angle and G is the universal gravitational constant [2-3]. These equations have heretofore constituted the basis of the study of the motions of planets around their stars and artificial satellites and projectile in the earth's atmosphere according to Einstein's Geometrical Theory of Gravitation known as General Relativity. But the earth and all the other planets as well as the Sun are now well-known experimentally as Spheroidal in shape [4-9].

Therefore, previous treatment of them as perfect sphere is at best an approximation for the sake of mathematical convenience. Moreover, the effect of spheroidal shapes of a body in the motion of test particles in its gravitational field will not corresponding to the spherical body. Therefore, the need arises to extend the theory of motion in the solar system from the fields of bodies of perfect spherical geometry to those of Spheroidal geometry. Professor S. X. K. Howusu, in 2004 obtained Einstein's Equation of motion in the gravitational field of an Oblate Spheroidal Body [10]. These equations are given as:

$0 = \ddot{t} + f_\eta \dot{t}\dot{\eta} - f_\xi \dot{t}\dot{\xi}$ (6)

and

$0 = \ddot{\eta} - \frac{c^2}{2} e^{G-F} F_\eta \dot{t}^2 - \frac{1}{2} G_\eta \dot{\eta}^2 - G_\xi \dot{\eta}\dot{\xi} + a^2\eta(1+\xi^2)e^G \dot{\phi}^2$ (7)

and

$0 = 0 = \ddot{\xi} - \frac{c^2}{2} e^{H-F} F_\xi \dot{t}^2 + \frac{1}{2} G_\eta \dot{\eta}^2 - H_\eta \dot{\eta}\dot{\xi} - a^2\xi(1+\eta^2)e^H \dot{\phi}^2$ (8)

and

$0 = \ddot{\phi} - \frac{2\eta}{1-\eta^2}\dot{\eta}\dot{\xi} + \frac{2\xi}{1+\xi^2}\dot{\xi}\dot{\phi}$ (9)

Equations (6) – (9) give the exact equations of motion for a test particle in the gravitational field of homogeneous oblate Spheroidal body according to Einstein's Geometrical Laws of the General Relativity.

In this paper, we are out to derive and solve the Einstein's geometrical equation of the motion for test particles in general form using Einstein's geometrical law of General relativity in the field of a homogenous prolate spheroidal massive body.

2.0 METHODS

2:1 EQUATION OF MOTION

The Einstein's Geometrical Gravitation Metric Tensor Exterior to a Homogeneous Prolate Spheroidal Body, in the Prolate Spheroidal Coordinates (η, ξ, ϕ, x^0) with origin at the centre of the body as [1]:

$g_{00}(\eta, \xi, \phi) = e^{-F}$ (10)

and

$g_{11}(\eta, \xi, \phi) = -e^{-G}$ (11)

and

$g_{22}(\eta, \xi, \phi) = -e^{-H}$ (12)

and

$g_{33}(\eta, \xi, \phi) = -a^2(1-\eta^2)(\xi^2-1)$ (13)

Then, the contravariant metric tensors are:

$g^{00}(\eta, \xi, \phi) = e^{F}$ (14)

and

$g^{11}(\eta, \xi, \phi) = -e^{G}$ (15)

and

$g^{22}(\eta, \xi, \phi) = -e^{H}$ (16)

and

$g^{33}(\eta, \xi, \phi) = -\frac{1}{a^2(1-\eta^2)(\xi^2-1)}$ (17)

where F, G and H are functions of the Prolate Spheroidal Coordinates $\eta and \xi$ only, and their equations are known[1]. But Einstein's Geometrical Equation of Motion in gravitational fields is given by:

$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$ (18)

where τ is the proper time and $\Gamma^\alpha_{\mu\nu}$ are the Riemann Christofel symbols [1], and x^0 is the coordinates of space-time. In the Prolate Spheroidal Coordinates with origin at the centre of the body

$x^0 = ct$(19)
$x^1 = \eta$ (20)
$x^2 = \xi$ (21)
$x^3 = \phi$(22)
where t is the coordinate of time and the connection coefficients are given by
$\Gamma^0_{10} = \Gamma^1_{01} = -\frac{1}{2}F_\eta$(23)
$\Gamma^0_{20} = \Gamma^1_{02} = -\frac{1}{2}F_\xi$(24)
$\Gamma^1_{00} = -\frac{1}{2}e^{G-F}F_\eta$(25)
$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{2}G_\xi$(26)
$\Gamma^1_{22} = -\frac{1}{2}e^{G-H}H_\eta$ (27)
$\Gamma^1_{33} = -a^2\eta(\xi^2 - 1)e^G$(28)
$\Gamma^2_{00} = -\frac{1}{2}e^{H-F}F_\xi$(29)
$\Gamma^2_{11} = -\frac{1}{2}e^{H-G}G_\xi$ (30)
$\Gamma^2_{12} = \Gamma^2_{21} = -\frac{1}{2}H_\eta$ (31)
$\Gamma^2_{22} = -\frac{1}{2}\mathrm{H}_\xi$ (34)

$\Gamma^2_{33} = -G^2\xi(1-n^2)e^H$ (35)

$\Gamma^3_{13} = \Gamma^3_{31} = -\frac{\eta}{1-\eta^2}$ (36)

$\Gamma^3_{23} = \Gamma^3_{32} = -\frac{\xi}{\xi^2-1}$ (37)

$\Gamma^\alpha_{\mu v} = 0, otherwise$ (38)

It therefore follows from (18) and (23) – (38) that;

$$0 = \ddot{t} - f_\eta \dot{t}\dot{\eta} - F_\xi \dot{t}\dot{\xi} \quad (39)$$

and

$$0 = \ddot{\eta} - \frac{1}{2}e^{G-f}F_\eta - G_\xi\dot{\eta}\dot{\xi} - \frac{1}{2}e^{G-H}H_\eta\xi^2 + a^2\eta(\xi^2 - 1)e^G \quad (40)$$

and

$$0 = \ddot{\xi} - \frac{c^2}{2}e^{H-F}f_\xi\dot{t}^2 - \frac{1}{2}e^{H-G}G_\xi\dot{\eta}^2 - H_\eta\dot{\xi}\dot{\eta} - \frac{1}{2}H_\xi\dot{\xi}^2 - a^2\xi\left(1-\eta^2 e^H\dot{\phi}^2\right) \quad (41)$$

And

$$0 = \ddot{\phi} - \frac{2\eta}{1-\eta^2}\dot{\eta}\dot{\phi} + \frac{2\xi}{\xi^2-1}\dot{\xi}\dot{\phi} \quad (42)$$

The results (39) – (42) are exact equations of motion for a fest particle in the gravitational field of homogenous Prolate Spherical body according to Einstein's Geometrical laws of General Relativity.

3.0 General Azimuthal Solution

Diving the azimuthal equation of motion (42) by $\dot{\phi}$ and integrating, it follows that;

$$\dot{\phi}^{(\eta,\xi,\phi)} = \frac{l}{\xi^2}(1-\eta^2)^{-1}\left(1-\frac{1}{\xi^2}\right)^{-1} \tag{43}$$

where l is a constant of motion defining the angular momentum per unit rest mass.

4.0 Solution in the Equatorial Plane

For motion confined to the equatorial plane of the body, we choose;

$$\eta \equiv 0 \tag{44}$$

Consequently the time equation (39) reduces to:

$$0 = \ddot{t} - f^1(\xi)\dot{\xi}\dot{t} \tag{45}$$

where

$$f(\xi) = \mathrm{F}(\eta,\xi)|_{\eta=0} \tag{46}$$

This equation is integrated exactly to yield;

$$\dot{t} = Aexp[-f(\xi)] \tag{47}$$

where A is an arbitrary constant. But by definition

$$\dot{t} \to 1 \quad as\xi \to \infty \tag{48}$$

and hence

$$A = exp[-f(\xi)] \tag{49}$$

In the second place, in the equatorial plane the equation of motion in the $\hat{\eta}$ direction (40) becomes

$$F_\eta(\eta,\xi)|_{\eta=0} \equiv 0 \tag{50}$$

In the third place in the equatorial plane the azimuthal solution (43) reduces to;

$$\dot{\phi} = \frac{l}{\xi^2}\left(1-\frac{1}{\xi^2}\right)^{-1} \tag{51}$$

In the fourth place in the equatorial plane, the equation of motion in the $\hat{\xi}$ direction (41) subject to (47) and (51) yields:

$$\ddot{\xi} = \frac{a^2l^2}{\xi^3}\left(1-\frac{1}{\xi^2}\right)^{-2} exp[h(\xi)] + \frac{1}{2}C^2A^2f(\xi)exp[f(\xi) + h(\xi)] \tag{52}$$

Where

$$H(\xi) = \mathrm{H}(\eta,\xi)|_{\eta=0} \tag{53}$$

We can say by principle that the results (52), (51) and (47) constitute a sufficient solution of motion in the equatorial plane in terms of the radial coordinateξ. But it may be most interesting and instructive to express the motion in terms of the angular coordinateϕ. Towards this goal let $w(\phi)$ be the reciprocal "distance" defined by:

$$w(\phi) = \frac{1}{\xi(\phi)} \tag{54}$$

Then it follows that

$$\ddot{\xi} = -l^2w^2(1-w^2)^{-1}\frac{d}{d\phi}\left[(1-w^2)^{-1}\frac{dw}{d\phi}\right] \tag{55}$$

Consequently, (52) becomes;

$$-l^2w^2(1-w^2)^{-1}\frac{d}{d\phi}\left[(1-w^2)^{-1}\frac{dw}{d\phi}\right] = a^2w^2l^2(1-w^2)^{-2}exp\left[h\left(\frac{1}{w}\right)\right] + \frac{1}{2}C^2A^2f\left(\frac{1}{w}\right)exp\left\{f\left(\frac{1}{w}\right) + h\left(\frac{1}{w}\right)\right\} \quad (56)$$

Or equivalently

$$\frac{d^2w}{d\phi^2} - 2w(1-w^2)^{-1}\frac{dw}{d\phi} = -a^2wexp\left\{h\frac{1}{w}\right\} - \frac{1}{2l^2}(1-w^2)^2C^2A^2.F\left(\frac{1}{w}\right)exp\left\{f\left(\frac{1}{w}\right) + h\left(\frac{1}{w}\right)\right\} \quad (57)$$

Equation (57) is the planetary equation in the equatorial plane of a homogeneous Prolate Spheroidal body according to Einstein's Geometrical Laws of General Relativity. It is therefore now opened up for comparison with the well known Einstein's Planetary Equation in the field of a homogeneous spherical massive body and a homogeneous oblate spheroidal massive body.

5.0 Summary and Conclusion

In this paper our Einstein's Geometrical Equation of Motion for a test particle in the gravitational fields of a homogeneous prolate spheroidal massive body are given by equations (39), (40), (41) and (42). Then we showed how they may be solved exactly and analytically for motion confined to the equatorial plane as equation (57).
It is most interesting and instructive to note that the planetary equation (57) contains at least one term of order w^2 corresponding to orbital precession.

It may be noted that the equation(57) obtained in this paper contains infinitely many pure spheroidal terms and hence the effects are now opened up for mathematical analysis and physical experimental and investigation in the motions of the planets, comets and asteroids in the solar system.

It may also be noted that equation (57) is now opened for the mathematical analysis of Einstein's equations of motion for test particle in the gravitational field exterior to a homogenous Prolate Spheroidal massive body for more general motions other than the one confined to the equatorial plane of the body.

Finally, equation (57) of Einstein's Geometrical equation of motion for a test particle in the gravitational field of homogeneous Prolate Spheroidal body derived in the paper may now be compared with the corresponding Newton's dynamical equation derived by Prof. S.X.K. Howusu, 2005 [11].

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