On Dynamic of a Novel Cubic Chaotic Map of the Plane

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Abstract: In the theoretical research of chaotic dynamical system, the different type of bifurcations is a very interesting powerful tool for analyzing the qualitative behavior of chaotic dynamical system, this short paper deals with the existence of symmetric chaotic attractors and some different type of bifurcations such as symmetry bifurcation, flip bifurcation, Hopf bifurcation and symmetry breaking bifurcation of a simple two-dimensional symmetry discrete chaotic map of the plane with cubic non-linearity. The dynamical behaviors of the map are investigated by mathematical analysis and simulated numerically using package of Matlab. We compute numerically the bifurcation diagram and largest Lyapunov exponent and phase portraits. The research results indicate that there are interesting nonlinear physical phenomena in this simple two-dimensional symmetry discrete cubic map of the plane, such as symmetry bifurcation, flip bifurcation, Hopf bifurcation, symmetry breaking bifurcation, and symmetric attractors. The important non-linear physical phenomena obtained in this paper would benefit the study of the cubic chaotic map and the development of the theory of chaotic discrete dynamical systems.

Keywords: Two-dimensional symmetry cubic map, flip bifurcation, symmetry-breaking bifurcation, Hopf bifurcation, symmetric attractors.

1. Introduction

Chaos or sensitive dependence on initial conditions, as a most fascinating phenomenon in non-linear dynamical systems has been intensively studied over the past few decades[1-2-10]. Chaos or chaotic behavior has been very useful in many fields. Many researches in continuous and discrete systems have been studied by this approach, such as biological systems[3-4-5], finance systems[6-7], applied sciences and engineering systems[8-9-10], secure communication and information processing systems[11]. Moreover however, there are many works that focus on the chaotic behavior of two-dimensional [12-13-14-15] and three-dimensional [16-17-18-19] discrete maps. Also known as, the study of chaotic behavior of cubic discrete map is a very interesting branch in dynamical systems such as for example in literature[20], H. R. Dullin and J. D. Meiss studied the dynamical behavior a new two-dimensional area-preserving cubic discrete map with spatial symmetry which displayed complicated behavior. This paper has reported a further investigation into a two-dimensional discrete cubic chaotic map with symmetry and rarely observed phenomenon: the existence of both, symmetric chaotic attractors and some different type of bifurcations such as symmetry bifurcation, flip bifurcation, Hopf bifurcation and symmetry breaking bifurcation. The new map generating complicated non-linear physical phenomena and capable of generating symmetric chaotic attractors from different initial conditions and through different bifurcation parameter values. Dynamical behavior has been reported within some bifurcation parameter values range. The fundamental dynamical behaviors, including largest Lyapunov exponents, bifurcation analysis and phase portraits are also simulated numerically when the bifurcation parameter varied to verify map behaviors.

2. The noval two dimensional cubic map

The general equation of the two-dimensional cubic map given by:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy + a_6 x^2 y + a_7 xy^2 + a_8 x^3 + a_9 y^3 \\ b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 y^2 + b_5 xy + b_6 x^2 y + b_7 xy^2 + b_8 x^3 + b_9 y^3 \end{pmatrix} \quad (1)$$

Where $(x, y) \in \mathbb{R}^2$ and $(a_i, b_i)_{0 \le i \le 9} \in \mathbb{R}^{20}$ are bifurcation parameters. The two-dimensional cubic maps are classified according to their number of nonlinearities and the first case of one cubic non-linearity is defined by:

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix} y \\ ay - ax^2y\end{pmatrix} \qquad (2)$$

Where x and y are the state variables of the map, and a is the bifurcation parameter. For $a = 0$, the map (2) reduces to a two-dimensional discrete linear map. The two-dimensional map (2) is defined by a linear function of one variable y and a non-linear function cubic of two variables x and y. The map (2) is defined for all points in the plane and the associated function of the map is of class $C^{\infty}(\mathbb{R}^2)$ and map (2) is symmetric under the coordinate transformation $(x, y) \to (-x, -y)$.

2.1 Local bifurcation results

Briefly, the fixed points of map (1) are:

$(0,0)$ and $\pm\left(\sqrt{\frac{a-1}{a}}, \sqrt{\frac{a-1}{a}}\right)$

and the Jacobi matrix of the map (2) evaluated at the fixed point (x, y) is:

$$J_{(x,y)} = \begin{pmatrix} 0 & 1 \\ -2axy & a - ax^2\end{pmatrix}$$

The characteristic polynomial of the Jacobi matrix $J_{(x,y)}$ is:

$$P(\lambda) = \lambda^2 - (a - ax^2)\lambda + 2axy$$

By criteria in[21], the trivial fixed point (0,0) of the map (2) is asymptotically stable if and only if the following conditions hold:

$$1 - a > 0, 1 + a > 0$$

or, equivalently,

$$-1 < a < 1$$

For example, if we choose $a = -0.2$ and $-(0.01,0.01)$ then with this value the fixed point $(0,0)$ is asymptotically stable and we have the following two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.7$ thus $|\lambda_{i(1\leq i\leq 2)}| < 1$ (see Fig. 1a). And if we choose $a = 0.7$ and $(0.01,0.01)$ then with this value the fixed point $(0,0)$ is asymptotically stable, and we have the following two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 0.7$ thus $|\lambda_{i(1\leq i\leq 2)}| < 1$ (see Fig. 1b).

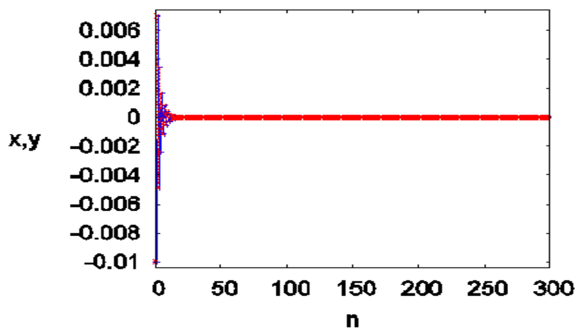


Fig. 1 a: The output time series x and y with $-(0.01, 0.01)$ and $a = -0.70$.

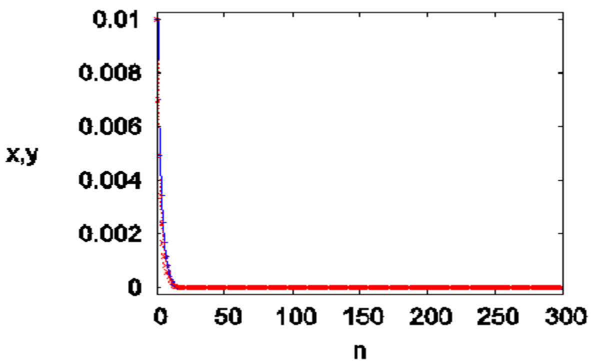


Fig. 1 b: The output time series x and y with $(0.01, 0.01)$ and $a = 0.70$.

Undoubtedly, the local stability of $(0,0)$ is studied by evaluating the eigenvalues of the Jacobi $J_{(0,0)}$.Than one have the following results:

(a) $|\lambda_2| < 1$, if and only if $-1 < a < 1$, map (2) is attracting at this fixed point.
(b) $|\lambda_2| > 1$, if and only if $a \in (-\infty, -1) \cup (1, +\infty)$, map (2) is a saddle at this fixed point.

By criteria in[21], the fixed points $\pm\left(\sqrt{\frac{a-1}{a}}, \sqrt{\frac{a-1}{a}}\right)$ of the map (2) are asymptotically stable if and only if the following conditions hold:

$$1 - 1 + 2(a-1) > 0,$$
$$1 + 1 + 2(a-1) > 0,$$
$$1 - 2(a-1) > 0$$

or, equivalently,

$$1 < a < \frac{3}{2}$$

For example, if we choose $a = 1.25$ and $(0.01, 0.01)$ then with this value the fixed points $\pm(0.44721, 0.44721)$ are asymptotically stable, and we have the following two eigenvalues $\lambda_1 = 0.5 - 0.5i$ and $\lambda_2 = 0.5 + 0.5i$ thus $|\lambda_{i(1\le i\le 2)}| <1$ (see Fig. 2a). And if we choose $a = 1.4$ and $(0.01, 0.01)$ then with this value the fixed points $\pm(0.53452, 0.53452)$ are asymptotically stable, and we have the following two eigenvalues $\lambda_1 = 0.5 - 0.74i$ and $\lambda_2 = 0.5 + 0.74i$ thus $|\lambda_{i(1\le i\le 2)}| <1$ (see Fig. 2b).

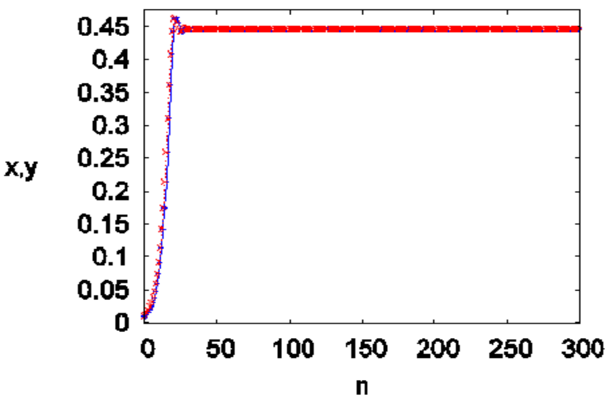


Fig. 2 a: The output time series x and y with $(0.01, 0.01)$ and $a = 1.25$.

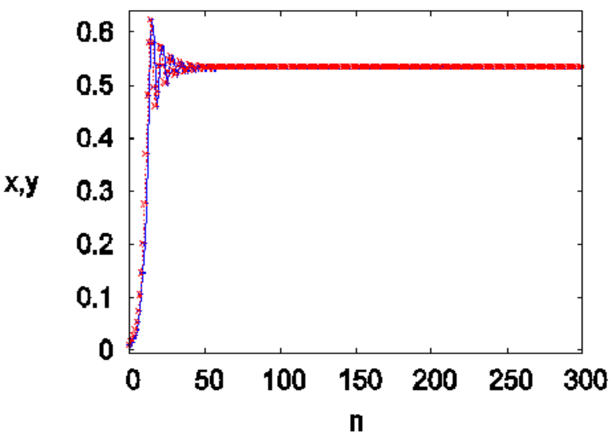


Fig. 2 b: The output time series x and y with $(0.01, 0.01)$ and $a = 1.40$.

Clearly, the local stability of points $\pm\left(\sqrt{\frac{a-1}{a}},\sqrt{\frac{a-1}{a}}\right)$ is studied by evaluating the eigenvalues of the Jacobi $J_{\pm\left(\sqrt{\frac{a-1}{a}},\sqrt{\frac{a-1}{a}}\right)}$.

If $a \in (-\infty,0) \cup (1,\frac{9}{8})$ the eigenvalues of $J_{\pm\left(\sqrt{\frac{a-1}{a}},\sqrt{\frac{a-1}{a}}\right)}$ are:

$\lambda_1 = \frac{1-\sqrt{9-8a}}{2}$ and $\lambda_2 = \frac{1+\sqrt{9-8a}}{2}$.

Than one have the following results:

(a) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $1 < a < \frac{9}{8}$, map (2) is attracting at this fixed point.
(b) $\lambda_1 = \lambda_2 = \frac{1}{2} < 1$ if and only if $a = \frac{9}{8}$, map (2) is attracting at this fixed point.
(c) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or ($|\lambda_1| < 1$ and $|\lambda_2| > 1$), impossible, map (2) is not a saddle at this fixed point.
(d) $|\lambda_1| > 1$ and $|\lambda_2| > 1$, if and only $a < 0$, map (2) is repelling at this fixed point.

If $a > \frac{9}{8}$ the eigenvalues of $J_{\pm\left(\sqrt{\frac{a-1}{a}},\sqrt{\frac{a-1}{a}}\right)}$ are:

$\lambda_1 = \frac{1-i\sqrt{-(9-8a)}}{2}$ and $\lambda_2 = \frac{1+i\sqrt{-(9-8a)}}{2}$.

Than one have the following results:

(a) $|\lambda_1| < 1$ and $|\lambda_2| < 1$, if and only if $\frac{9}{8} < a < \frac{3}{2}$, map (2) is attracting at this fixed point.
(b) $\lambda_1 = \lambda_2 = \frac{1}{2} < 1$ if and only if $a = \frac{9}{8}$, map (2) is attracting at this fixed point.
(c) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or ($|\lambda_1| < 1$ and $|\lambda_2| > 1$), impossible, map (2) is not a saddle at this fixed point.
(c) $|\lambda_1| > 1$ and $|\lambda_2| > 1$, if and only $a > \frac{3}{2}$, map (2) is repelling at this fixed point.

2.2 Bifurcation and attractors results

In this section, the dynamical behaviors of the map (2) are obtained through numerical computation and simulations. There are several possible ways for a discrete map to make a transition from regular behavior to chaotic behavior. Bifurcation diagrams display these routes to determine the long-time behavior and chaotic zones, we numerically compute the bifurcation diagrams and largest Lyapunov exponent in the interval $[-1.99,1.99]$. Figures 3 and 4 shows respectively, the bifurcation diagram and the Lyapunov exponent spectrum diagram of map (2). Figures 5 and 6 illustrate respectively some observed of attractors generated from the two symmetrical $\pm(0.01,0.01)$ initial conditions and for some symmetrical $\pm a$ bifurcation parameters values. Figures 5 represents the attractors generated from the negative $-(0.01,0.01)$ initial condition and for some negative bifurcation parameter values a. Figures 6 represents the attractors generated from the positive $(0.01,0.01)$ initial condition and for some positive bifurcation parameter values a. The left column attractors and chaotic attractors in Figures 5 contact with each other at the origin in the quadrant 2 and 4 of the x-y plane, which are symmetric about the origin. Similarly, the right column attractors and chaotic attractors in Figures 6 contact with each other at the origin in the quadrant 1 and 3 of the x-y plane, which are symmetric about the origin.

(a) $-1.99 \le a < -1.50$, map (2) is chaotic (LEs >0, see Fig. 5c, Fig. 5d, Fig. 5e and Fig. 5f), periodic windows exists in the chaotic zone (LEs ≤0, Fig. 5a and Fig. 5b).
(b) $a = -1.50$, the Hopf bifurcation occurs at this point (see Fig. 4a).
(c) $-1.50 < a < -1$, the map (2) is period-2 attractor (LEs <0, see Fig. 4a).
(d) $a = -1$, the Flip bifurcation occurs at this point (see Fig. 4a).
(e) $-1 < a < 0$, map (2) is period-1 (LEs <0, see Fig. 4a).
(f) $0 \le a < 1$, map (2) is period-1 (LEs <0, see Fig. 3a).

(g) $a = 1$, the symmetry-breaking bifurcation occurs at this point. If the initial condition is $(0.01,0.01)$, the first branch occurs (see Fig. 3a), if the initial condition is $-(0.01,0.01)$, the second branch occurs (see Fig. 4a).
(h) $1 < a < 1.50$, map (2) is period-1 attractor (LEs <0, see Fig. 3a).
(i) $a = 1.50$, the Hopf bifurcation occurs in the map (2) (see Fig. 3a).
(j) $150 < a < 1.99$, map (2) is chaotic (LEs >0, see Fig. 6c, Fig. 6d, Fig. 6e and Fig. 6f), periodic windows exists in the chaotic zone (LEs ≤0, Fig. 6a and Fig. 6b).

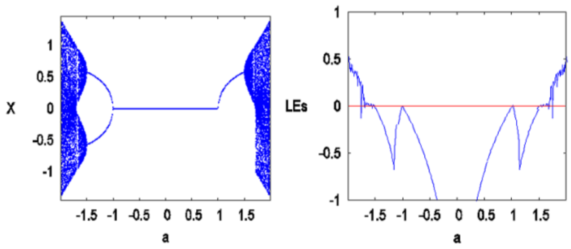


Fig. 3 a: Bifurcation diagram and Fig. 3b: Lyapunov exponent for $\boldsymbol{a} \in [-1.99, 1.99]$ with $(0.01, 0.01)$.

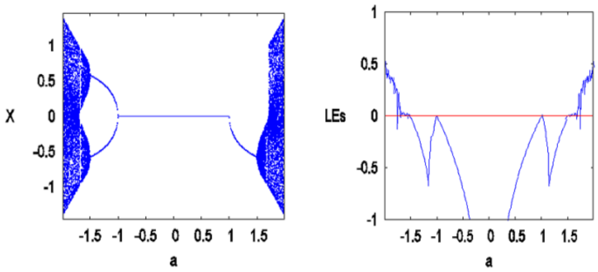
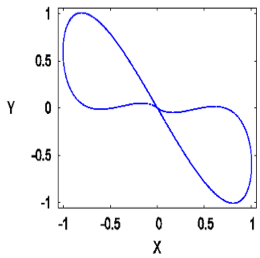


Fig. 4 a: Bifurcation diagram and Fig. 4b: Lyapunov exponent for $\boldsymbol{a} \in [-1.99, 1.99]$ with $-(0.01, 0.01)$.



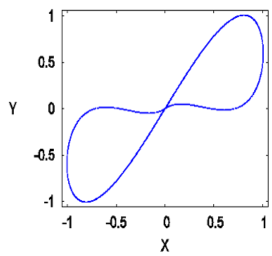
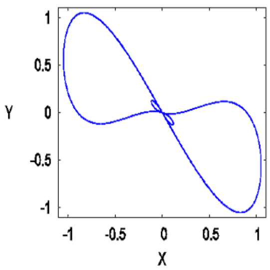


Fig. 5 a: a = −1.716

Fig. 6 a: a = 1.716



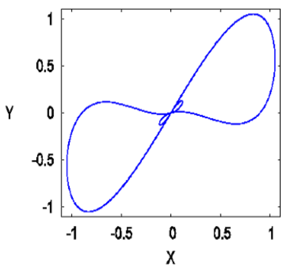


Fig. 5 b: a = −1.75

Fig. 6 b: a = 1.75

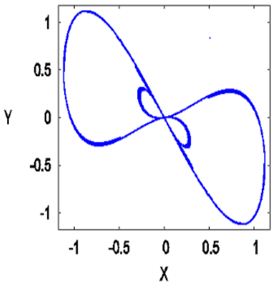


Fig. 5 c: a = −1.80

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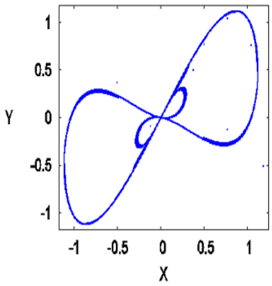

Fig. 6 c: a = 1.80

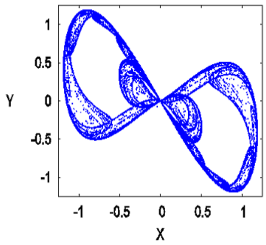

Fig. 5 c: a = −1.85

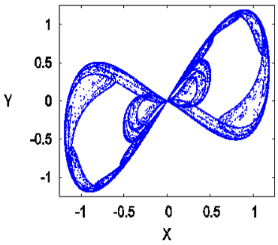

Fig. 6 c: a = 1.85

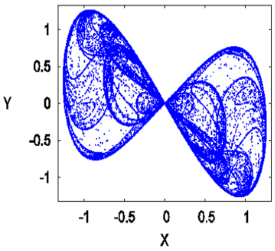

Fig. 5 d: a = −1.90

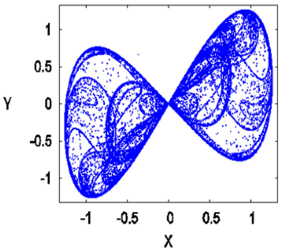

Fig. 6 d: a = 1.90

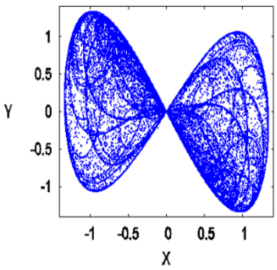

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Fig. 5 e: a = −1.95

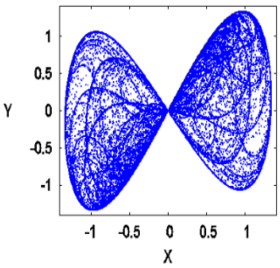

Fig. 6 e: a = 1.95

3. Conclusin

This short paper deals with the existence of both symmetric chaotic attractors and some different type of bifurcations such as symmetry bifurcation, flip bifurcation, Hopf bifurcation and symmetry breaking bifurcation of a simple two-dimensional symmetry discrete chaotic cubic map of the plane. Dynamical behavior has been reported within some bifurcation parameter values. The basic dynamical behaviors, including stability, Lyapunov exponents spectrum, bifurcation analysis and phases portrait are also illustrated to verify map behaviors.

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